JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 32, No. 2, March–April 2009

Engineering Notes

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Minimality of Variable-Thrust Subarcs in Optimal Chemical Rocket Trajectories

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DOI: 10.2514/1.38443

I. Introduction

FROM the beginning of rocket trajectory optimization, it has been known that the optimal trajectory of a rocket can be composed of three types of subarcs: maximum thrust, variable thrust, and zero thrust. To determine whether or not a particular subarc can be part of a minimum-fuel trajectory, it is possible to apply some tests (necessary conditions). Necessary conditions for a relative minimum are classified according to the types of the controls, that is, regular controls (nonsingular), singular controls, and mixed controls (regular and singular), and according to level, that is, weak or strong. Weak conditions are the Legendre–Clebsch condition [1] for regular controls, the generalized Legendre–Clebsch condition [2] for singular controls, and Goh's condition [3] for mixed controls. Strong conditions are the Weierstrass condition [1] for regular controls, the equivalent of the Weierstrass conditon [4] for singular controls, and the equivalent of Goh's condition for mixed controls (yet to be derived).

A subarc is defined to be "minimizing" if the relevant necessary conditions for a minimum are satisfied. Although this use of the term minimizing might seem to be unconventional to some researchers, it is not. See for example Bliss [5] (p. 20), Bryson and Ho [6] (p. 247), Kopp and Moyer [7] (p. 1443), Goh [3] (p. 721), and Bell and Jacobson [2] (p. 3).

For a chemical rocket, the particular variable-thrust subarc, known as Lawden's spiral, was claimed by Lawden to be minimizing by applying the Weierstrass condition [8] (p. 112). Subsequently, Robbins [9], Kelley [10] , Kopp and Moyer [7], and Kelley et al. [11] developed the generalized Legendre–Clebsch condition and applied it to the Lawden spiral. Their conclusion was that the Lawden spiral is not minimizing. The same conclusion was reached by Kelley [12] using state transformations. Later, Lawden [13] wrote that the variable-thrust subarc is not minimizing but did not say why the Weierstrass condition gave a conflicting result.

This Note has two purposes. The first is to review the preceding necessary conditions and their use. The second is to explain why Lawden's application of the Weierstrass condition did not eliminate the variable-thrust subarc.

Presented as Paper 413 at the AAS/AIAA Astrodynamics Specialist Conference, Mackinac Island, MI; received 7 May 2008; revision received 24 October 2008; accepted for publication 27 October 2008.

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II. Regular Controls

Let u_k denote a component of the control vector u. A regular control u_k is a control where the optimality condition $H_{u_k} = 0$ explicitly contains the control u_k, so that $H_{u_k u_k}$ is not identically zero. The Weierstrass condition is a necessary condition for a regular optimal control (Ewing [14] p. 64) to be a strong relative minimum. In the development of the Weierstrass condition for unbounded controls ([1], Chap. 10), it is assumed that a solution of the first-order optimality conditions is known and that it is to be tested for minimality. The solution is a local minimum if the total change in the augmented performance index from the comparison control to the optimal control satisfies the relation $\Delta J' > 0$ regardless of the choice of the comparison control. The particular comparison control used to derive the Weierstrass condition is a strong pulse from t_p to t_q followed by a weak variation from t_q to t_f that satisfies the final conditions. By assuming that $\Delta t = t_q - t_p$ is small, it can be shown that

$$\Delta J' = [H(t, x, u*, \lambda) - H(t, x, u, \lambda]_p \Delta t + O(\Delta t^2) > 0 \quad (1)$$

Here, the state $x(n \times 1)$, the control $u(m \times 1)$, and the multiplier $\lambda(n \times 1)$ are given by the optimal solution, and u_* is the arbitrary magnitude of the pulse. Because the time t_p is arbitrary, the Weierstrass condition is given by

$$\Delta H = H(t, x, u*, \lambda) - H(t, x, u, \lambda) \geq 0 \quad (2)$$

at every point of the optimal solution. The Weierstrass condition can be applied to each control separately or to any combination. For bounded controls, the minimum principle ([6], p. 108) requires that u_* satisfy the bounds.

Actually, $\Delta H = 0$ when $u_* = u$. For $u_* \neq u$, it must be that $\Delta H > 0$. Otherwise, there is a neighboring control (nonoptimal), which gives the same value of J'. In other words, for the optimal control to be a minimizing control, u must be an absolute minimum of H with respect to the control ([6], p. 108).

The Legendre–Clebsch condition is the weak form of the Weierstrass condition. It is obtained by assuming a weak pulse, that is, $u_* = u + \delta u$ and expanding Eq. (2) in a Taylor series. For independent controls, the result is

$$H_{uu} \geq 0 \quad (3)$$

The Legendre–Clebsch condition can be applied to the controls individually or in any combination. To achieve $H_{uu} > 0$, u must be a local minimum of H with respect to the control.

If a particular optimal control u_k is such that $H_{u_k u_k} = 0$ on u_k, higher derivatives of H with respect to u_k can be used to examine minimality. An example of this occurs in the problem defined by $\min J = \int u^4 \,\mathrm{d}t$, $\dot{x} = u$, $t_0 = 0$, $x_0 = 0$, $t_f = 0$, $x_f = 0$. Here, the optimal control is $u = 0$. Then, on $u = 0$, $H_{uu} = 0$, $H_{uuu} = 0$, and $H_{uuuu} = 24$, so that $u = 0$ satisfies the weak necessary condition for a minimal control . On the other hand, if a particular control u_k is such that $H_{u_k u_k} \equiv 0$, all higher derivatives of H with respect to u_k are identically zero, so that the optimal control is singular.

III. Singular Controls

A singular control u_k occurs when $H_{u_k u_k} \equiv 0$. The most important case is when the Hamiltonian contains one or more control variables linearly. In this case, the condition $H_{u_k} = 0$ does not explicitly contain the control u_k so that $H_{u_k u_k} \equiv 0$.

Consider the dynamical system

$$\dot{x} = p(t, x) + q(t, x)u \tag{4}$$

If the performance index does not contain the control, the Hamiltonian is written as

$$H = \lambda^T(p + qu) \tag{5}$$

so that the application of the optimality condition $H_u = 0$ yields the result

$$\lambda^T q = 0 \tag{6}$$

which does not explicitly contain the control. Hence, $H_{uu} \equiv 0$ and the control is singular.

Next, the Weierstrass condition in Eq. (2) becomes

$$\Delta H = \lambda^T q(u_* - u) \equiv 0 \tag{7}$$

Because Eq. (1) with $\Delta H \equiv 0$ does not yield a condition, the Weierstrass condition does not apply to a singular control.

For this problem,

$$H_{uu} \equiv 0 \tag{8}$$

so that the Legendre–Clebsch condition does not apply to a singular control. The equivalent of the Legendre–Clebsch condition for a singular control is the generalized Legendre–Clebsch condition [2,11]

$$(-1)^q \frac{\partial}{\partial u}\left(\frac{\mathrm{d}^2 q}{\mathrm{d}t^2 q} H_u\right) \geq 0 \tag{9}$$

where q denotes the order of the singular control. For $q = 1$, the derivation of Eq. (9) is based on a comparison control, which is a weak double pulse (doublet) followed by a weak variation that satisfies the final conditions. For $q = 2$, the comparison control comes from two double pulses. If the left-hand side of Eq. (9) is positive for $q = 1$, the optimal singular control is minimizing. If the left-hand side of Eq. (9) is zero, the $q = 2$ condition must be investigated.

The equivalent of the Weierstrass condition for a $q = 1$ singular control has been developed in [4] using a strong double pulse. It is shown that the strong condition is identical with the weak condition (9). In other words, the results of [4] elevate the status of the generalized Legendre–Clebsch condition from a weak condition to a strong condition for $q = 1$. It is anticipated that the same will be true for $q = 2$.

IV. Mixed Controls

Consider an optimal control problem involving both regular and singular controls. The Weierstrass condition and the Legendre–Clebsch condition can be applied to individual or combinations of regular controls, and the generalized Legendre–Clebsch condition can be applied to individual singular controls. However, the complete proof of weak minimality requires the use of Goh's condition [3] The equivalent of Goh's condition for a strong relative minimum is not yet available.

V. Optimal Trajectories for Chemical Rockets

It is desired to find the controls of a chemical rocket that maximize the final mass during a transfer from an initial state to a final state for flight in a plane, in a vacuum, and in a central force field. For a chemical rocket, $T = \beta c$ and $\dot{m} = -\beta$ where T is the thrust, β is the controlled propellant mass flow rate, c is the constant exhaust velocity, and m is the mass of the rocket. These equations can be rewritten with thrust as the control as $\beta = T/c$ and $\dot{m} = -T/c$. Then, by integrating the mass equation, the minimum-fuel problem is to find the control histories that minimize the performance index

$$J = \int_{t_0}^{t_f} \tau \, \mathrm{d}t \tag{10}$$

subject to the differential constraints

$$\begin{aligned} \dot{r} &= v_r \\ \dot{\theta} &= v_\theta / r \\ \dot{v}_r &= v_\theta^2/r - \mu/r^2 + \tau \sin\phi \\ \dot{v}_\theta &= -v_r v_\theta / r + \tau \cos\phi \end{aligned} \tag{11}$$

subject to prescribed boundary conditions, and subject to control inequality constraints. In the differential equations, r is the radial distance of the rocket from the center of the planet, θ is the polar angle of the position vector, v_r is the radial component of velocity, and v_θ is the transverse component of velocity. The controls are the thrust per unit mass, that is, $\tau = T/m$, and the angle ϕ between the thrust direction and the local horizontal.

The inequality constraint is on the thrust magnitude. In this analysis, only the variable-thrust subarc is being investigated. Hence, the controls are off the boundary and are independent.

In this formulation, the thrust is singular, and the thrust angle is regular. One way to examine the minimality of the optimal solution is to assume that the optimal singular control is fixed and apply the necessary conditions to the regular control. Then, the optimal regular control is fixed, and the necessary conditions are applied to the singular control. Finally, both controls should be varied simultaneously.

In the solution of the optimal control problem, use is made of the variational Hamiltonian H. That part of H that contains the controls is given by

$$H = \tau + \cdots + \lambda_3 \tau \sin\phi + \lambda_4 \tau \cos\phi \tag{12}$$

Application of the control conditions leads to

$$\begin{aligned} H_\tau &= 0\colon\ 1 + \lambda_3 \sin\phi + \lambda_4 \cos\phi = 0 \\ H_\phi &= 0\colon\ \tau(\lambda_3 \cos\phi - \lambda_4 \sin\phi) = 0 \end{aligned} \tag{13}$$

These equations yield two solutions: the coast subarc $\tau = 0$ and the variable-thrust subarc.

For the variable-thrust subarc, Eq. (13) leads to

$$\lambda_3 = -\sin\phi, \qquad \lambda_4 = -\cos\phi \tag{14}$$

Next, it is seen that

$$\begin{aligned} \Delta H = H(t, x, u_*, \lambda) - H(t, x, u, \lambda) = \tau_* + \lambda_3 \tau_* \sin\phi_* \\ + \lambda_4 \tau_* \cos\phi_* - \tau - \lambda_3 \tau \sin\phi - \lambda_4 \tau \cos\phi \end{aligned} \tag{15}$$

Then, because of Eqs. (13) and (14), the equation for ΔH becomes

$$\Delta H = \tau_*[1 - \cos(\phi_* - \phi)] \tag{16}$$

Let the singular control be held at its optimal value, that is, $\tau_* = \tau$, and vary the regular control. The application of the Weierstrass condition leads to

$$\Delta H = \tau[1 - \cos(\phi * - \phi)] \geq 0$$

which is satisfied. The regular control satisfies the necessary condition for a strong relative minimum.

Next, let the regular control be held at its optimal value, that is, $\phi_* = \phi$, and vary the singular control. Because $\Delta H \equiv 0$, the Weierstrass condition and the Legendre–Clebsch condition are not applicable. The generalized Legendre–Clebsch condition is available for testing weak minimality. So far, this condition has only been applied to the variable-thrust subarc known as Lawden's spiral. Lawden's spiral is a second-order singular control, and it has been found not to be minimizing [11].

For Lawden's spiral, it is not necessary to vary both controls simultaneously (mixed controls) because it has already been shown not to be minimizing. However, to do so requires the use of Goh's condition for weak minimality and the equivalent of Goh's condition for strong minimality. Hence, the reason the Weierstrass condition and the generalized Legendre–Clebsch condition gave conflicting results is that the Weierstrass condition by itself is not enough to prove that Lawden's spiral is minimizing.

Lawden's spiral can only be a part of a minimum-fuel trajectory for the boundary conditions of free final time and fixed final polar angle. There are other classes of solutions for different boundary conditions, and none of these has yet been examined for minimality.

VI. Conclusions

The purpose of this Note is to review the necessary conditions for a relative minimum and to apply them to the variable-thrust subarc of a minimum-fuel rocket trajectory known as Lawden's spiral.

The necessary conditions are discussed for regular controls, singular controls, and mixed controls. Within each of these categories, the conditions are classified as strong or weak. The Weierstrass condition and the Legendre–Clebsch condition, which are valid for regular controls, are discussed and shown not to be applicable to singular controls. On a singular control, weak minimality can be tested by applying the generalized Legendre–Clebsch condition. There exists a strong condition, equivalent to the Weierstrass condition, but it has only been derived for a first-order singular control. This condition has the same formula as the generalized Legendre–Clebsch condition. Hence, the status of the generalized Legendre–Clebsch condition has been elevated from a weak condition to a strong condition in this case.

For a chemical rocket, previous analyses of the particular variable-thrust subarc known as Lawden's spiral showed that the Weierstrass condition and the generalized Legendre–Clebsch condition gave conflicting results. This optimal control problem is characterized by one regular control (thrust angle) and one singular control (thrust). By varying just the singular control (thrust), the Weierstrass condition and the Legendre–Clebsch condition do not apply, but the generalized Legendre–Clebsch condition indicates that Lawden's spiral is not minimizing. If the thrust and thrust direction are varied simultaneously, Goh's condition must be applied. In other words, the Weierstrass condition by itself is not enough to prove that Lawden's spiral is minimizing. This explains why the Weierstrass condtion and the generalized Legendre–Clebsch condition gave conflicting results.

Acknowledgment

The author is indebted to Dilmurat Azimov for sharing his knowledge of singular optimal control problems and optimal rocket trajectories.

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